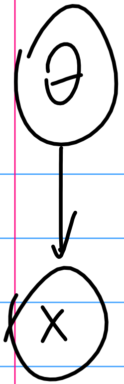


$\sim \mathrm{Dir}(\alpha_1, \ldots, \alpha_k) \Rightarrow \vec{\theta} = \{\theta_1, \ldots, \theta_k\}; \ \sum \theta_i = 1$

$$P(\vec{\theta}) = P(\theta_1, \ldots, \theta_k) = \frac{1}{Z} \prod_{i=1}^{k} \theta_i^{\alpha_i - 1}$$

$X \sim \mathrm{Multinomial}(\theta_1, \ldots, \theta_k)$

$$Z = \frac{\prod_{i=1}^{k} \Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)}$$

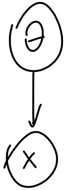
$$P(X) = \int P(X|\vec{\theta}) P(\vec{\theta}) d\vec{\theta}$$

$$P(X = x_i) = \frac{1}{Z} \int_{\vec{\theta}} \theta_i \prod_j \theta_j^{\alpha_i - 1} d\vec{\theta}$$

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du$$

$$= \frac{\alpha_i}{\sum_j \alpha_j}$$

Why ?



$\theta \sim Dir(\alpha_1, \dots, \alpha_k)$

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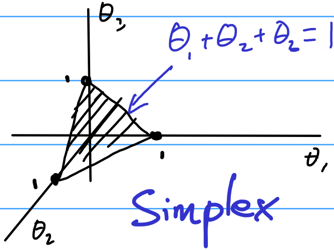
$$Z = \frac{\prod_{i=1}^{k}\Gamma(\alpha_i)}{\Gamma(\sum_{i=1}^{k}\alpha_i)}$$

$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}du$$

$k=3$: $\vec{\theta} = \{\theta_1, \theta_2, \theta_3\}$; $P(X=x_3) = ?$

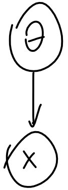
$$\int_{\theta_1=0}^{1}\int_{\theta_2=0}^{1-\theta_1}\cancel{\int_{\theta_3}} \theta_1^{\alpha_1-1}\,\theta_2^{\alpha_2-1}\,\underbrace{\theta_3^{\alpha_3}}_{(1-\theta_1-\theta_2)}\,d\theta_1\,d\theta_2\,\cancel{d\theta_3}$$

Limits of integration?



Simplex

1. Draw a simplex
2. Write $\int$ limits in a clever way to make the whole thing a k-fold Laplace convolution
3. ???
4. PROFIT!!!



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$k = 3$: $\vec{\theta} = \{\theta_1, \theta_2, \theta_3\}$; $P(X = x_3) = ?$

Limits of integration?

$$\int_{\theta_1=0}^{\infty} \int_{\theta_2=0}^{\infty} \int_{\theta_3=0}^{\infty} \theta_1^{\alpha_1 - 1}\, \theta_2^{\alpha_2 - 1}\, \theta_3^{\alpha_3}\, \underbrace{\delta(1 - \theta_1 - \theta_2 - \theta_3)}_{\begin{cases} 1 & \text{if } 1 - \theta_1 - \theta_2 - \theta_3 = 0 \\ & \left(\text{i.e. } \sum \theta_i = 1\right) \\ 0 & \text{otherwise} \end{cases}}\, d\theta_1\, d\theta_2\, d\theta_3$$

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx}\, dk$$

$$\int_{\theta_1=0}^{\infty}\int_{\theta_2=0}^{\infty}\int_{\theta_3=0}^{\infty} \theta_1^{\alpha_1-1}\,\theta_2^{\alpha_2-1}\,\theta_3^{\alpha_3}\,\delta\big(\underbrace{1-\theta_1-\theta_2-\theta_3}\big)\,d\theta_1\,d\theta_2\,d\theta_3$$

$$Z=\frac{\prod_{i=1}^{k}\Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}$$

$$\Gamma(x)=\int_0^{\infty}u^{x-1}e^{-u}\,du$$

$$\delta(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{\underbrace{-ik(1-\theta_1-\theta_2-\theta_3)}_{}}\,dk$$

$$e^{-ik}\,e^{ik\theta_1}\,e^{ik\theta_2}\,e^{ik\theta_3}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ik}\int_{\theta_1=0}^{\infty}\theta_1^{\alpha_1-1}e^{ik\theta_1}\,d\theta_1\int_{\theta_2=0}^{\infty}\theta_2^{\alpha_2-1}e^{ik\theta_2}\,d\theta_2\int_{\theta_3=0}^{\infty}\theta_3^{\alpha_3}e^{ik\theta_3}\,d\theta_3\,dk$$

$$ik := -\kappa$$

$$k = i\kappa$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\kappa t}\int_{\theta_1=0}^{\infty}\theta_1^{\alpha_1-1}e^{-\kappa\theta_1}\,d\theta_1\int_{\theta_2=0}^{\infty}\theta_2^{\alpha_2-1}e^{-\kappa\theta_2}\,d\theta_2\int_{\theta_3=0}^{\infty}\theta_3^{\alpha_3}e^{-\kappa\theta_3}\,d\theta_3\,d\kappa\,\Big|_{t=1}$$

$ik := -\kappa$

$k = i\kappa$

$$Z = \frac{\prod_{i=1}^{k}\Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}$$

$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}\,du$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\kappa t}\,\mathcal{L}\{\theta_1^{\alpha_1-1}\}\,\mathcal{L}\{\theta_2^{\alpha_2-1}\}\,\mathcal{L}\{\theta_3^{\alpha_3}\}\,d\kappa\,\Big|_{t=1}$$

OMG Laplace transform!!!

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,ds$$

$$\int_0^\infty \theta_3^{\alpha_3}e^{-s\theta_3}\,d\theta_3 = \int \frac{s^{\alpha_3}\theta_3^{\alpha_3}e^{-s\theta_3}\,d(\underbrace{s\theta_3}_{u})}{s^{\alpha_3}\cdot s}$$

$$= \int_0^\infty \frac{1}{s^{\alpha_3+1}}\,u^{(\alpha_3+1)-1}e^{-u}\,du = \frac{\Gamma(\alpha_3+1)}{s^{\alpha_3+1}}$$

$$\mathcal{L}\{t^{\alpha}\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\kappa t}\int_{\theta_1=0}^{\infty}\theta_1^{\alpha_1-1}e^{-\kappa\theta_1}d\theta_1\int_{\theta_2=0}^{\infty}\theta_2^{\alpha_2-1}e^{-\kappa\theta_2}d\theta_2\int_{\theta_3=0}^{\infty}\theta_3^{\alpha_3}e^{-\kappa\theta_3}d\theta_3\,d\kappa\Bigg|_{t=1}$$

$ik := -\kappa$

$k = i\kappa$

$Z = \frac{\prod_{i=1}^{k}\Gamma(\alpha_i)}{\Gamma(\sum_{i=1}^{k}\alpha_i)}$

$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}du$

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\kappa t}\,\mathcal{L}\{\theta_1^{\alpha_1-1}\}\,\mathcal{L}\{\theta_2^{\alpha_2-1}\}\,\mathcal{L}\{\theta_3^{\alpha_3}\}\,d\kappa\Bigg|_{t=1}$$

OMG Laplace transform!!!

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}ds$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\kappa t}\,\frac{\Gamma(\alpha_1)}{\kappa^{\alpha_1}}\,\frac{\Gamma(\alpha_2)}{\kappa^{\alpha_2}}\,\frac{\Gamma(\alpha_3+1)}{\kappa^{\alpha_3+1}}\,d\kappa\Bigg|_{t=1}$$

$\Gamma(\alpha_3+1) \to \alpha_3\Gamma(\alpha_3)$

$\mathcal{L}\{t^\alpha\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$

$\Gamma(n) = (n-1)! \Rightarrow \Gamma(n+1) = n\Gamma(n)$

$$\alpha_3\prod\Gamma(\alpha_i)\,\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\kappa t}\,\frac{1}{\kappa^{1+\alpha_1+\alpha_2+\alpha_3}}\,d\kappa\Bigg|_{t=1}$$

$$\alpha_3 \prod \Gamma(\alpha_i)\, \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{kt}\, \frac{1}{k^{1+\alpha_1+\alpha_2+\alpha_3}}\, dk \,\Bigg|_{t=1} = \alpha_3 \prod_i \Gamma(\alpha_i)\, \mathcal{L}^{-1}_{k\to t}\left\{\frac{1}{k^{1+\sum\alpha_i}}\right\}$$

$$Z = \frac{\prod_{i=1}^{k}\Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}$$

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u}\, du$$

inverse

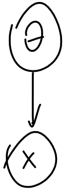
OMG ^ Laplace transform !!!

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} F(s)\, ds$$

$$t^{\alpha} = \mathcal{L}^{-1}\left\{\mathcal{L}\{t^{\alpha}\}\right\} = \mathcal{L}^{-1}\left\{\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}\right\} = \Gamma(\alpha+1)\,\mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha+1}}\right\} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{k^{1+\sum\alpha_i}}\right\} = \frac{t^{\sum\alpha_i}}{\Gamma\left(1+\sum\alpha_i\right)}$$

$$\alpha_3 \prod_i \Gamma(\alpha_i)\, \mathcal{L}^{-1}_{k\to t}\left\{\frac{1}{k^{1+\sum\alpha_i}}\right\}\Bigg|_{t=1} = \frac{\alpha_3 \prod_i \Gamma(\alpha_i)}{\Gamma\left(1+\sum\alpha_i\right)} = \frac{\alpha_3}{\sum_{i=1}^{3}\alpha_i}\, \underbrace{\frac{\prod_i \Gamma(\alpha_i)}{\Gamma\left(\sum\alpha_i\right)}}_{Z}$$

$$\Gamma\left(1+\sum\alpha_i\right) \hookrightarrow \left(\sum\alpha_i\right)\Gamma\left(\sum\alpha_i\right)$$



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$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}\,du$$

$$k=3: \quad \vec{\theta} = \{\theta_1, \theta_2, \theta_3\}\ ; \quad P(X=x_3) = \cancel{\frac{1}{Z}}\, \frac{\alpha_3}{\sum_{i=1}^{3}\alpha_i}\, \cancel{Z}$$

$$\boxed{P(X=x_j) = \frac{\alpha_j}{\sum_{i=1}^{k}\alpha_i}}$$

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\,dk + \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,ds + \Gamma(x) = \int_0^\infty u^{x-1}e^{-u}\,du$$

$$= \mathrm{WIN}.$$

The End :)